

XXXIX. *A Letter to the Right Honourable Hugh Earl of Marchmont, F. R. S. concerning the Sections of a Solid, hitherto not considered by Geometers; from William Brakenridge, D. D. Rector of St. Michael Bassishaw London, and F. R. S.*

My Lord,

Read Dec. 20, 1759. **Y**OUR knowledge in Geometry, and the other Sciences that depend upon it, makes me presume to lay before you the following speculations. Your benevolence to all philosophical Inquiries encourages me, and the personal regard I have for your Lordship induces me to do myself this honour; and tho' what I offer at present may be of no great consequence, I am perswaded, that every little accession to our knowledge will give you some pleasure, as you very well know, that all our improvements in science are slow, and from small beginnings. You have here a new method of considering some geometrical curves, from the sections of a solid, hitherto not taken notice of, and by which, in particular, you will see, that the two infinite curve lines from the section of the cone, are also the sections of this; which may be of some use, as it seems to extend our views of their nature and properties. The description of it is very easy and obvious, and it has something remarkable in its form, that tho', in the most simple case, it is generated and bounded by right lines, the surface is incurvated. The solid is thus described.



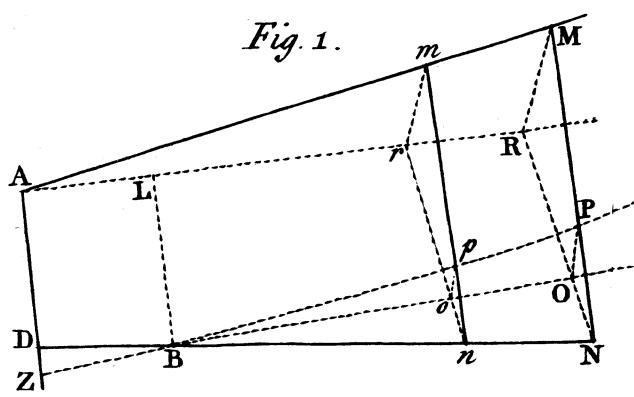


Fig. 1.

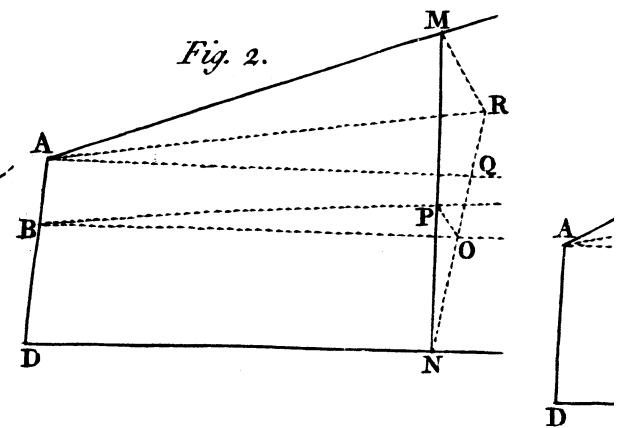


Fig. 2

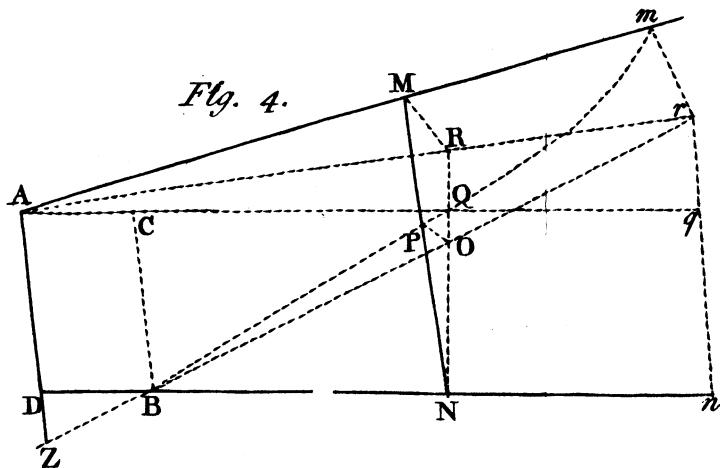


Fig. 4.

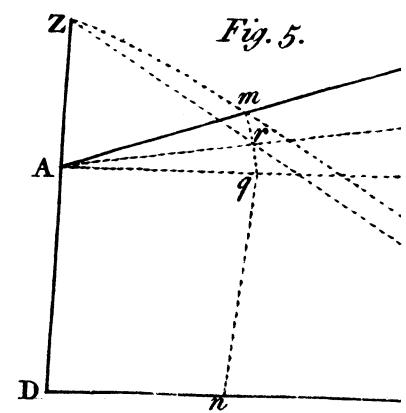


Fig. 5.

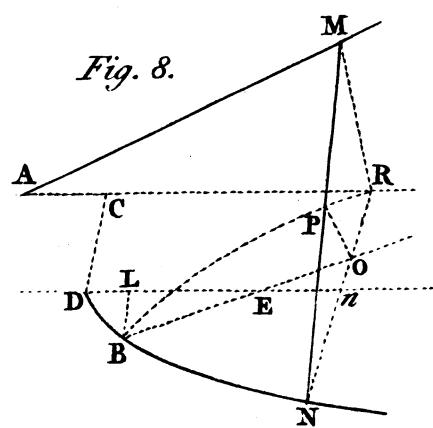


Fig. 8.

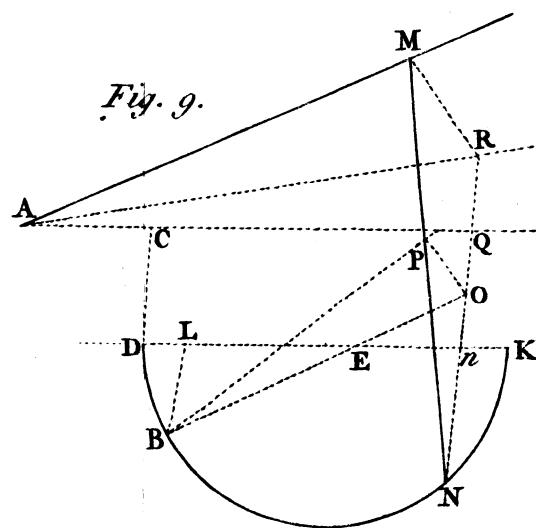


Fig. 9.

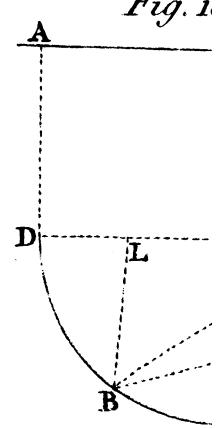


Fig. 1.

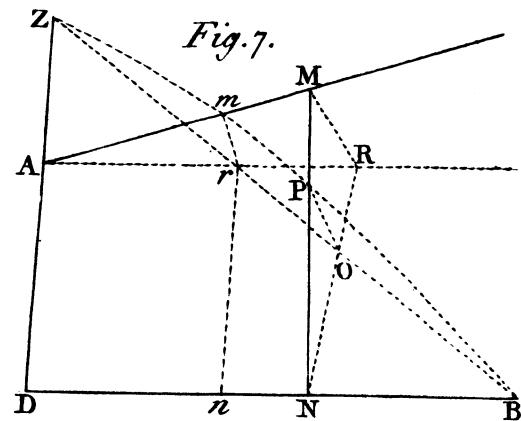
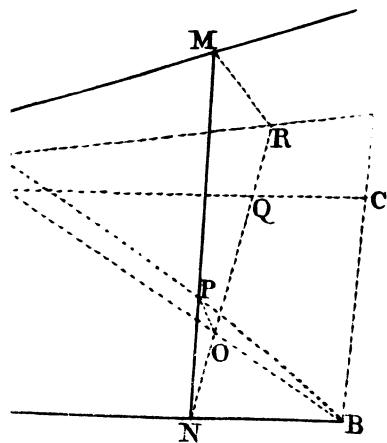
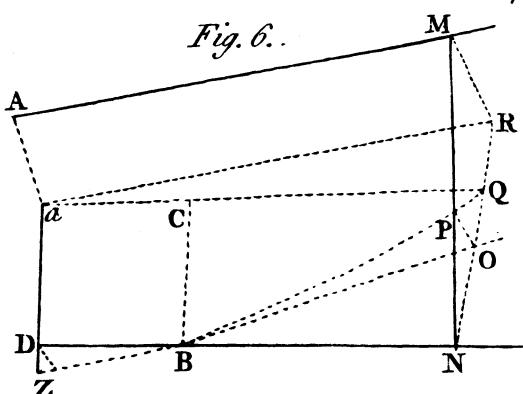
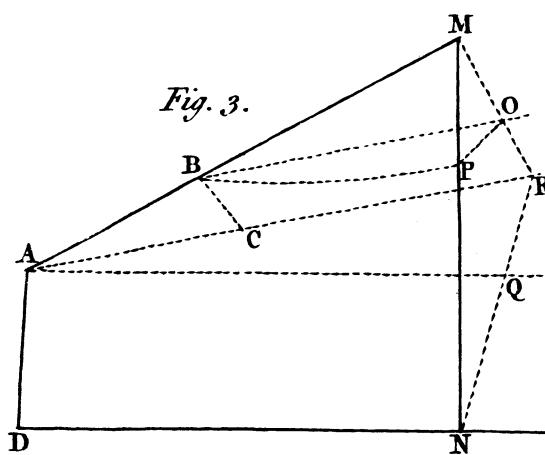
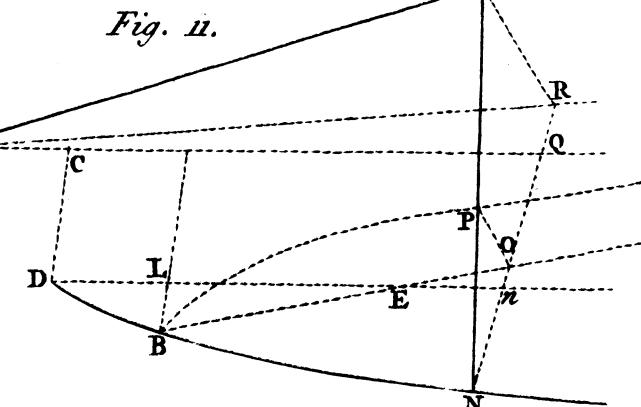
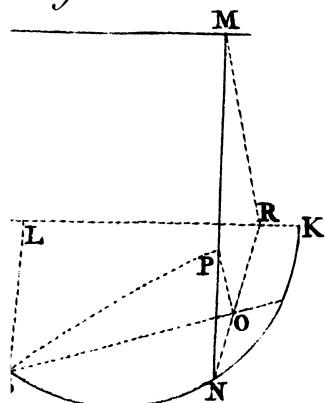


Fig. 10.



Let DN be a right line drawn on a plane, and a point given A at any distance from the line, from which having raised, above the plane, the right line AM in any given angle, and drawing from the same point A the line AD , meeting the line DN on the plane in D , make another plane MNR to move parallel to itself, and to that line AD in a given angle to the first plane; and then, suppose the intersections of it $M, N, m, n, \&c.$ with the lines AM and DN , to be continually joined with right lines $MN, mn, \&c.$ and there will be generated an incurvated surface by them, and bounded by the lines AM, MN, ND, AD . *Plate X. Fig. 1.*

The same surface will be described, if a line MN be supposed to move in such manner, along the lines AM and DN , that the parts AM and DN be continually in a given ratio. For let a plane be drawn thro' the raised line AM , perpendicular to the given plane ADN , and their common intersection be the line AR ; then having joined the points A and D , from the point N , while the line MN is moving, let there be continually drawn a line NR parallel to DA , meeting the line of intersection AR of the planes in R , and the proportion of DN to AR will always be given. But as the ratio of DN to AM is given, the ratio of AM to AR will also be given; and the line RM being drawn, the angle ARM will be given; and therefore the plane of the triangle MRN will move parallel to itself, and to the line AD , making the given angle MRA with the plane ADN , and the line MN will generate the same surface as before.

It is evident this incurvated surface may be infinitely extended on all sides beyond the lines A M, M N, N D, A D, and as well below the given plane A D N as above it; and therefore the various sections of it, if continued, will be infinite lines.

The line A M, raised above the given plane A D N, may be called the vertical Directrix.

And if thro' this directrix A M there be drawn a plane A M R in any given angle to the plane A D N, intersecting the moving plane N M R in the line M R, and meeting the given plane A D N in R; then the points A, R, and R, N, being joined, there will be a trapezium A R N D, formed, that may be named the Base, and in which the line D N may be called the Directrix of the Base.

And the plane A M R passing thro' the vertical directrix and the moving plane M N R, together with the base A R N D, and the curve surface, will make a solid A M R N D A, something in the form of a wedge.

In this solid there may be made five sections in a given angle with the base, or parallel to it; viz, one parallel to the moved plane; one parallel to the directrix of the base; another parallel to the vertical directrix; a fourth parallel to the base; and a fifth intersecting both the directrices. And in all these cases, when the directrices are right lines, the sections will be either the conic hyperbola, or the parabola, or right lines.

1st, If the solid A M R N D A be cut by a plane parallel to the moved or generating plane N M R, or the line A D, the section will be a right line. This is evident from the description, *Fig. 1.*

2. If the section be made by a plane B P O parallel to the directrix of the base D N, and in any given angle to the base, it will be an hyperbola, *Fig 2.*

Let the section P B O meet the base in the line B O, which will be parallel to D N, and make the moving plane N M R intersect the base in N R, the vertical directrix in M, and the section in P O; by which D B=N O. From the point M draw M R parallel to P O, and imagine a plane to pass through the directrix A M and the line M R, meeting the base in A R; which will be given in position. Then thro' the point A draw A Q, parallel to D N, intersecting N R in Q, and making N Q=A D; and the two triangles A R Q, A M R, will have all their angles given, and the proportion of their sides. And therefore the ratio of A Q to Q R, and of A Q to M R, will be given. Make A D=a, B D=b, A Q:Q R :: a:q, and A Q: M R :: a:m, the abscisse B O=x, and the ordinate P O=y; from which $Q R = \frac{q x}{a}$,

$N R = a + \frac{q x}{a}$, $M R = \frac{m x}{a}$. And then from the similar triangles N O P, N M R, the analogy N O:O P :: N R: M R will give the equation $y x q + y a^2 = x b m$, and the curve B P is an hyperbola.

3. If the section P B be made by a plane parallel to the vertical directrix A M, it will be an hyperbola. *Fig. 1.*

Let the moving plane M R N, parallel to D A, intersect the base in R N, and the vertical directrix in M, and make the plane of the section B P O cut the moving plane N M R in P O, and the base in the line B O, meeting D A in Z, and the base directrix D N in B; from the point M draw the line

MR parallel to the plane of the section, and meeting the base in R; and the lines MR and PO will be parallel. Then thro' the vertical AM, and the line MR, make a plane to pass, which will be parallel to the section, intersecting the base in AR, and the lines AR and BO will also be parallel. Draw BL parallel to DA, or NR, passing AR in L, and then BO=LR, BL=RO=AZ, and AL=BZ. And the line AR being given in position, the lines BL and AL will be given. The angles also MAR and NBO being given, the ratio of AR to RM, and of BO to ON, will be given. Suppose AD=x=BL=a, AL=c the abscisse BO=x, the ordinate PO=y, and AR:RM::a:m, and BO:ON::a:n, from which we have ON= $\frac{nx}{a}$, RM= $\frac{mc+mx}{a}$;

NR=a+ $\frac{nx}{a}$. And then, in the triangles on the moving plane, which are similar, MRN, PON, the analogy NO:PO::NR:RM will produce the equation $y^2a^3+nx^2a^2y=x^2n^2m+xnmc$; which shews the curve BP to be an hyperbola; and the figure is convex to the base.

4. The section being made parallel to the base, it will be the same curve. *Fig. 3.*

Having thro' the vertical directrix made a plane AMR to pass perpendicular to the base, let the section BPO, and the base parallel to it, meet that plane in the lines BO and AR, and also the moving plane NMR in the lines PO and NR. From the point A draw AQ parallel to the directrix DN of the base, and AD parallel to NR, and from B the line BC parallel to OR, meeting AR in C; and then

then the lines AD , AC , and BC , will be given, and the angles RAQ , MAR , being also given, the proportion of AR to QR , and of BO to MO , will be known. Make $AD=a$, $AC=c$, the abscisse $BO=x$, the ordinate $PO=y$, and $AR:RQ::a:q$; from which we have $MO:MR::x:x+c$, $QR=\frac{cq+qx}{a}$. And because, in the similar triangles MPO , and MNR , in the moving plane, we have $MO:PO::MR:RN$, there will result the equation $yx a + yc a = x^2 q + xc q + x a^2$; which denotes the curve BP to be an hyperbola.

5. If the section is made so as to meet the two directrices, the curve will be also an hyperbola. *Fig. 4.*

Let the section BPO meet the directrices in B, m , and intersect the plane of the base in the line BO ; and make the moving plane NMR to be cut by the section in PO , and to meet the vertical directrix in M . Then from the point M draw MR parallel to PO ; and thro' the lines AM, MR , imagine, as before, a plane to pass, intersecting the base in the line ARr , and meeting the line BO in r , and the section in mr . And from A draw AD parallel to NR , and AQ to DN ; and thro' r make nr parallel to NR , and to meet AQ in q , and DN in n . Draw also from the point B the line BC parallel to AD , and meeting AQ in C . The lines then $Aq, Ar, mr, rq, Br, Bn, rn$, will be given. Make therefore $AD=a=BC$, $AQ:QR::a:q$, and $AQ:RM::a:m$, $Br:Bn::a:b$, $Br:rn::a:n$, $AC=DB=c$; the abscisse $BO=x$; the ordinate $OP=y$. From which we have $BN=\frac{bx}{a}$, $ON=\frac{nx}{a}$, $AQ=c+\frac{bx}{a}$.

$\frac{bx}{a}$, $RQ = \frac{qca + qb^x}{a^2}$, $MR = \frac{mca + mb^x}{a^2}$. And then, in the similar triangles N P O, M R N, having $NO : PO :: NR : MR$, the equation will be $ya^4 + yca^2q + yxb^2a^2q = mncax + mb^2x^2$. And the curve is an hyperbola; and in the case of this *Fig. 4.* it will be convex to the plane of the base. But when BN is negative in the case of *Fig. 5.* the equation, retaining the same symbols, will be $ya^4 + yqca^2 - yxb^2a^2q = mncax - mb^2x^2$; and the hyperbolic curve will be concave towards the base.

6. If the vertical directrix AM is made parallel to the plan of the base, but the plane passing thro' it not parallel to the other directrix, then the section, meeting the two directrices, will also be the same curve. *Fig. 6.*

For in this case the line MR is a constant quantity; and therefore, if the common section aR of the plane thro' the vertical AM, with the base, meet aD parallel to NR in a, other being as before, making $MR = m$, and $Da = a$; from the analogy $NO : PO :: NR : MR$ in the triangles NOP, MRN, we shall have $mncax = ya^3 + yacq + yxb^2q$; by which the curve is known to be an hyperbola.

And in all those sections, where the common intersection of the plane, passing thro' the vertical with the base, is not parallel to the other directrix, the curve is an hyperbola.

7. But now, if we suppose the common intersection aR of the plane passing thro' the vertical, with the base, to be parallel to the directrix DN of the base, and both directrices to be cut by the section, the curve will be a parabola, *Fig. 7.*

For

For, in this case, the two lines $A R$ and $A Q$ coincide, and $A R$ is parallel to the directrix $D N$ of the base ; and therefore, using the same symbols as above, the equation, from *Fig. 4.* will be reduced to $y a^4 = n m c a x + n m b x^2$; and from *Fig. 5.* it will come to $y a^4 = n m c a x - n m b x^2$; both which shew the curve to be a parabola.

It may also be demonstrated in this manner. It is evident, that $P O : m r :: P O \times M R : m r \times M R$, that is, the ratio of $P O$ to $m r$ is equal to that compounded of $P O : M R$, and of $M R : m r$. But from the similar triangles $M R N$, $P O N$, we have $P O : M R :: N O : N R = n r$, and from the triangles $B O N$, $B r n$, we have $B O : B r :: N O : n r$; therefore $P O : M R :: B O : B r$. In like manner, from the equiangular triangles $M R A$, $m r A$, there will be $M R : m r :: R A : r A$, and from the triangles $R O r$, $Z A R$, it is $R A : r A :: O z : r z$; therefore $M R : m r :: O z : r z$. If then in the ratio of $P O \times R M$ to $m r \times M R$, we substitute the ratio of $B O : B r$, and of $O z : r z$, which are equal to $P O : M R$, and $M R : m r$, we shall have $P O : m r :: B O \times O z : B r \times r z$; which is a known property of the parabola.

And thus I have endeavoured to extend, a little, the Theory of the conic sections. I have here shewn how two of them may be had from the sections of this solid ; and in the year 1733, I published, in my *Exercitatio Geometrica*, a method of describing all of them on a plane, by the moving of three right lines about three given points, two of the intersections being drawn along two other lines given in position ; the only hint of which I had from a geometric

tric locus, in the construction of a clock, by the celebrated M. De la Hire, in the French *Mémoires* for the year 1717, intituled, *Construction d'une Horologe qui marque le vrai tems avec le moyen.*

Having now briefly considered the various sections of this solid, the directrix D N of the base being a right line, let us next see what they are when it is a curve line of any order. But because there are an infinite number of cases, it will be sufficient to mention a few of them, when that directrix is a conic section or circle, and then to give a general proposition when it is any higher geometrical curve.

8. If the directrix D N of the base be a parabola, having a diameter parallel to the intersection A R of the plane, passing thro' the vertical A, and other things being as before, the section B P will be a line of the third order. *Fig. 8.*

Let the diameter be D L n , the ordinate N n , and the equation of the parabola $u^2 = zp$, and make the moving plan M N R to pass thro' the ordinate N n , and the section to intersect the base in B O, and the diameter in E. From the points B and D draw BL and D C, parallel to N O, and meeting the lines A R and D n in C and L. Then suppose A R : R M :: a : m , and B O : L n :: a : n , A C = c , D C = d , B L = b , D L = l , B E = e , N n = u , D n = z , the abscisse B O = x , the ordinate P O = y ; and we have L n = z - l = $\frac{nx}{a}$, $n o$ = $\frac{b \times x - e}{e}$, M R = $\frac{mc + mz}{a} = \frac{mca + mal + nxm}{a^2}$ and $u^2 = \frac{apl + pxn}{a}$. But

from the similar triangles N P O and N M R we have this proportion N O : P O :: N R : M R, which gives the

the equation $\frac{bx - be \times mca + mal + mnx - dy a^2 e}{ya^2 e - mca e - mal e - mnxe} = u = \sqrt{\frac{pl + pxn}{a}}$; and this being reduced, shews, that the section P B is a line of the third order.

9. If the directrix D N be a circle, and other things being as before, the section will be a line of the fourth order. *Fig. 9.*

Make the center of the circle to be in the line D L n parallel to A Q, the ordinate to be N n = u, the abscisse D n = z, the radius = r, and the equation $u^2 = 2r z - z^2$. And let the plane passing thro' the vertical, cut the base in A R, and the section meet the diameter D K in E. Then the same things being supposed, and the symbols retained as before, we shall have L n = $z - l = \frac{x n}{a}$, $z = \frac{al + xn}{a}$, $n O = \frac{bx - eb}{e}$, $Q R = \frac{qac + gal + qnx}{a^2}$, $M R = \frac{mac + mal + mnx}{a^2}$, $\sqrt{\frac{2rla^2 + 2rxna - x^2n^2 - 2alnx - a^2l^2}{a^2}} = u$. And the analogy $NO : PO :: NR : MR$ will give this equation, $\frac{mac + mal + mnx}{a^2} \times \frac{be - bx}{e} + \frac{ya^2d + yqac + yqal + yqnx}{a^2} = \frac{mac + mal + mnx - a^2y}{a^2}$ $\times \sqrt{\frac{2rla^2 + 2rxna - x^2n^2 - 2alnx - a^2l^2}{a^2}}$; from which it appears, that the curve B P is a line of the fourth order.

And in general it may be seen, that if the directrix of the base be a conic section, except in the case above, the section of the solid will be a line of the fourth order.

10. If the vertical be parallel to the base, and the plane passing thro' it perpendicular, the directrix of the base being a circle, having its center in the intersection A R of the two planes; then the solid will be the cono-cuneus of the learned Dr. Wallis, and the various curve sections of it will be also lines of the fourth order. *Fig. 10.*

In this case, the quantities Q R and D C will vanish, and making M R = m , the equation, retaining the other symbols, will be $m - y \times \sqrt{\frac{2rxna + 2rla^2 - x^2 - 2xnl a - a^2 l^2}{a^2}} = m \times \frac{eb - xb}{e}$.

And here it is surprising that the great Doctor, while he was considering his solid, did not fall upon the one I have explained; but indeed, in searching after new discoveries, we are often like those, who, groping in the dark, miss the things that are nearest them.

11. To conclude, if the directrix D N of the base be a line of any order n , the section B P will be of the order $2n$. *Fig. 11.*

In the equation of the curve directrix D N of the base $u^n = A z^n + B z^{n-1} u + C z^{n-2} u^2 + D z^{n-3} u^3$ &c. make the abscisse D $n = z$, and the ordinate N $n = u$; and draw A Q parallel to D n, and then, other things being as before, the analogy N O : P O :: N R : M R will be thus expressed, $u + \frac{bx - eb}{e} : y :: u + d + \frac{qnx + qal + qac}{a^2} : \frac{nmx + mal + mca}{a^2}$;

from which we shall have $u =$

$$\frac{eb - bx \times nmx + mca + mal}{emal + emnx + mace - a^2 ey} + \frac{da^2 y + nqxy + qaly + qyac}{mnx + mac - a^2 y + mal}.$$

And because D n = $z = l + \frac{nx}{a}$, if we substitute these values of u and z , in the above general equation,

the line of the section BP will appear to be of the order 2^n .

And now, my Lord, I have given you some general propositions about the various sections of this solid, and I have shewn how lines of any order may arise from them ; which is a new theory, and perhaps may introduce to something farther. I have other things of this sort, that relate to what I have published in the Philosophical Transactions, Vol. 39. in the year 1735, which I have had many years by me, that I intend to send you, if my health, and the circumstances of my life, will allow me to revise them. And in the mean time, with great respect, I am,

My Lord,

Your Lordship's most obedient
and affectionate servant,

Sion College, Dec. 18,
1759.

W^m. Brakenridge.

END *of* PART I.

